ON THE BASIC PROPERTIES OF A SYSTEM OF HOMOGENEOUS SOLUTIONS

PMM Vol. 31, No. 5, 1967, pp. 861-869
I. I. VOROVICH and V. E. KOVAL'CHUK
(Rostov-on-the-Don)
(Received February 21, 1967)

We investigate the first fundamental problem of the theory of elasticity for a rectangle. Assuming that its solution is a function whose fourth derivatives are summable with degree $p > 2$ over a rectangle, we prove that this solution can be expanded into a series in eigenfunctions of the problem considered, the series converging uniformly within the rectangle. We also establish how the series converges on the boundary and show that the expansion is unique.

1. Let us consider the first fundamental problem of the theory of elasticity for the rectangle $ABCD$ (Fig. 1). We assume that stresses are absent along BC and AD (they could always be removed by solving a problem for a strip). The stress function U of the investigated problem is biharmonic in the rectangle $ABCD$ and should therefore satisfy the following conditions: $U_{xx} = U_{xy} \equiv 0$ on BC and AD, $U_{yy} = \varphi_-(y)$ and $U_{xy} = \psi_-(y)$ on AB, $U_{yy} = \varphi_+(y)$ and $U_{xy} = \psi_+(y)$ on CD. We shall also assume that conditions of equilibrium of the rectangle

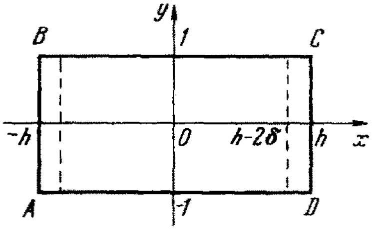

Fig. 1

$$\int_{-1}^{1} (\varphi_+ - \varphi_-)\, dy = 0, \qquad \int_{-1}^{1} (\psi_+ - \psi_-)\, dy = 0$$

$$\int_{-1}^{1} (\varphi_+ - \varphi_-)\, y\, dy + 2h \int_{-1}^{1} \psi_+\, dy = 0 \tag{1.1}$$

hold. Selecting out of the stress function U the factor corresponding to the unbalanced part of the stresses on AB and CD, we introduce the function

$$U_0 = \frac{1}{4} y^2 \int_{-1}^{1} \varphi_+\, ds + \frac{1}{4} y^3 \int_{-1}^{1} \varphi_+ s\, ds -$$

$$- \frac{3}{4} y (h - x) \int_{-1}^{1} \psi_+\, ds + \frac{1}{4} y^3 (h - x) \int_{-1}^{1} \psi_+\, ds$$

and put $u = U - U_0$.

Obviously, u satisfies the following boundary conditions:

$$u_{xx} = u_{xy} \equiv 0 \quad \text{on } BC \text{ and } AD \tag{1.2}$$

On CD we have

$$u_{yy} = \varphi_+(y) - \frac{1}{2} \int_{-1}^{1} \varphi_+\, ds - \frac{3}{2} y \int_{-1}^{1} \varphi_+ s\, ds = \Phi_+(y) \tag{1.3}$$

$$u_{xy} = \psi_+(y) - \frac{3}{4}\int_{-1}^{1} \psi_+ \, ds \, (1 - y^2) = \Psi_+(y) \tag{1.4}$$

from which we easily derive

$$\int_{-1}^{1} \Phi_+ \, dy = 0, \qquad \int_{-1}^{1} \Phi_+ y \, dy = 0, \qquad \int_{-1}^{1} \Psi_+ \, dy = 0 \tag{1.5}$$

which shall be used later. On AB we have

$$u_{yy} = \varphi_-(y) - \frac{1}{2}\int_{-1}^{1} \varphi_+ \, ds - \frac{3}{2} y \int_{-1}^{1} \varphi_+ s \, ds - 3hy \int_{-1}^{1} \psi_+ \, ds = \Phi_-(y) \tag{1.6}$$

$$u_{xy} = \psi_-(y) - \frac{3}{4}\int_{-1}^{1} \psi_+ \, ds \, (1 - y^2) = \Psi_-(y) \tag{1.7}$$

Relations (1.1), (1.6) and (1.7) together yield

$$\int_{-1}^{1} \Phi_- \, ds = 0, \qquad \int_{-1}^{1} \Phi_- s \, ds = 0, \qquad \int_{-1}^{1} \Psi_- \, ds = 0 \tag{1.8}$$

Now we shall find the values of u, u_x and u_y on the contour $ABCD$. The function u is defined with accuracy of up to a linear term, therefore we can assume that at some point A, say, $u = u_x = u_y = 0$. By (1.6) and (1.7) we then have

$$u_y = \int_{-1}^{y} \Phi_- \, ds, \quad u_x = \int_{-1}^{y} \Psi_- \, ds = f_3(y) \quad \text{on} \quad AB \tag{1.9}$$

At the point B we have, by (1.8), $u_x = u_y = 0$, therefore from (1.2) it follows that

$$u_x = u_y \equiv 0 \quad \text{on} \quad BC \tag{1.10}$$

Further we have on CD

$$u_y = \int_{1}^{y} \Phi_+ \, ds, \quad u_x = \int_{1}^{y} \Psi_+ \, ds = f_1(y) \tag{1.11}$$

By (1.11) and (1.5) we have, at D, $u_x = u_y = 0$, consequently

$$u_x = u_y \equiv 0 \quad \text{on} \quad AD \tag{1.12}$$

Thus first derivatives of u are defined along the whole contour $ABCD$. Further we

$$u = \int_{-1}^{y} u_y \, ds = \int_{-1}^{y}\int_{-1}^{s} \Phi_- \, ds_1 \, ds = \int_{-1}^{y} (y - s) \Phi_- \, ds = f(y) \quad \text{on } AB \tag{1.13}$$

At the point B we have, by (1.8), $u = 0$ and

$$u \equiv 0 \tag{1.14}$$

on BC in accordance with (1.10). On CD we have

$$u = \int_{1}^{y} u_y \, ds = \int_{1}^{y}\int_{1}^{s} \Phi_+ \, ds_1 \, ds = \int_{1}^{y} (y - s) \Phi_+ \, ds = f_2(y) \tag{1.15}$$

and at D we have $u = 0$ by (1.5). Finally, from (1.12) it follows that

$$u \equiv 0 \quad \text{on } AD \tag{1.16}$$

Thus (1.9) to (1.16) yield the following boundary conditions for the biharmonic function u

$$u|_{y=\pm 1} = 0, \qquad u_y|_{y=\pm 1} = 0$$

$$u|_{x=-h} = f(y), \quad u|_{x=h} = f_2(y), \quad u_x|_{x=-h} = f_3(y), \quad u_x|_{x=h} = f_1(y)$$

If the load of the plate is symmetric in x, we can put $f_2(y) = f(y)$, $-f_3(y) = f_1(y)$, and obtain, finally,

$$\Delta^2 u = 0; \quad u|_{y=\pm 1} = 0, \quad u_y|_{y=\pm 1} = 0, \quad u|_{x=\pm h} = f(y), \quad u_x|_{x=\pm h} = \pm f_1(y) \tag{1.17}$$

2. Problems of the type (1.17) are usually solved using the method of separation of varaibles, and solution of (1.17) is therefore sought in the form

$$u = \sum_k c_k a_k(y) \cos \lambda_k x \tag{2.1}$$

where $a_k(y)$ are eigenfunctions of the problem

$$a^{IV} - 2\lambda^2 a'' + \lambda^4 a = 0, \qquad a|_{y=\pm 1} = 0, \qquad a'|_{y=\pm 1} = 0$$

A requirement that the boundary conditions should be satisfied, leads to expansions

$$f(y) = \sum_k c_k \cos \lambda_k h a_k(y), \qquad f_1(y) = -\sum_k c_k \lambda_k \sin \lambda_k h a_k(y) \tag{2.2}$$

Thus, solution of (1.17) reduces to the problem of expanding a pair of arbitrary functions $f(y)$ and $f_1(y)$ into series (2.2).

This problem was first posed by Papkovich [1] and its general form is: to find a simultaneous expansion of two mutually independent real functions $f_1(y)$ and $f_2(y)$ into series of the form

$$f_1(y) = \sum_k c_k L_1[a_k(y)], \qquad f_2(y) = \sum_k c_k L_2[a_k(y)]$$

Here c_k are complex coefficients identical in both expansions, L_1 and L_2 are two distinct linear operators whose form depends on the actual problem, and $a_k(y)$ are the eigenfunctions of this problem.

In the present paper we develop a novel approach to investigations of problems of the type (1.17). We first obtain a solution of (1.17) in the form (2.1) inside the rectangle $ABCD$ and use this to obtain a solution of (2.2).

3. Let the fourth derivatives of the solution u of (1.17) be summable over the rectangle $ABCD$ with degree $p > 2$. (Question of restrictions which must be imposed on f and f_1 in order for the required condition to hold, demands a separate investigation and shall not be considered here).

Let us extend u to a strip $|y| \leq 1$. To do this, we shall use an even smoothing function $g(x)$, $|x| \leq h$ equal to unity when $|x| \leq h - \delta$, $0 < \delta < h/2$, infinitely differentiable and vanishing with all its derivatives at $|x| = h$, and we shall also put

$$u_1(x, y) = \begin{cases} u(x, y)\, g(x) & (|x| \leqslant h) \\ 0 & (|x| > h) \end{cases}$$

Function $u_1(x, y)$ defined in this manner on the strip $|y| \leq 1$, is a solution of the problem

$$\Delta^2 u_1 = \varphi(x, y); \qquad u_1|_{y=\pm 1} = 0, \quad u_{1y}|_{y=\pm 1} = 0 \tag{3.1}$$

where the function

$$\varphi(x, y) = \Delta^2 (ug) = 4(u_{xxx} + u_{xyy}) g' + 2(3u_{xx} + u_{yy}) g'' + 4u_x g''' + ug^{IV} \quad (3.2)$$

differs from zero only when $h - \delta < |x| < h$ and is an even function of x.

Applying the Fourier transformation in x to (3.1), we reduce it to an ordinary differential Eq.

$$(u^*)^{IV} - 2\lambda^2 (u^*)'' + \lambda^4 u^* = \varphi^*(y, \lambda), \qquad u^*|_{y=\pm 1} = 0, \quad (u^*)'|_{y=\pm 1} = 0 \quad (3.3)$$

where

$$u^*(y,\lambda) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u_1(x, y) e^{-i\lambda x} dx, \quad \varphi^*(y,\lambda) = \frac{2}{\sqrt{2\pi}} \int_{h-\delta}^{\infty} \varphi(x, y) \cos \lambda x \, dx \quad (3.4)$$

Differential operator (3.3) has an enumerable set of simple eigenvalues, which satisfy the Eq.

$$D(\lambda) = (\lambda + \mathrm{sh}\lambda\mathrm{ch}\lambda)(\lambda - \mathrm{sh}\lambda\mathrm{ch}\lambda) = 0 \quad (\lambda \neq 0) \quad (3.5)$$

Corresponding eigenfunctions are as follows:

$$a_k^{(1)}(y) = \lambda_k^{(1)} y \,\mathrm{sh}\, \lambda_k^{(1)} y + \mathrm{sh}^2 \lambda_k^{(1)} \mathrm{ch}\, \lambda_k^{(1)} y, \quad \text{if} \quad \lambda_k^{(1)} + \mathrm{sh}\, \lambda_k^{(1)} \mathrm{ch}\, \lambda_k^{(1)} = 0 \quad (3.6)$$

$$a_k^{(2)}(y) = \lambda_k^{(2)} y \,\mathrm{ch}\, \lambda_k^{(2)} y - \mathrm{ch}^2 \lambda_k^{(2)} \mathrm{sh}\, \lambda_k^{(2)} y, \quad \text{if} \quad \lambda_k^{(2)} - \mathrm{sh}\, \lambda_k^{(2)} \mathrm{ch}\, \lambda_k^{(2)} = 0 \quad (3.7)$$

We note that Eq. (3.5) and the root λ_k always have associated roots $-\lambda_k, \bar{\lambda}_k$ and $-\bar{\lambda}_k$, and, that the roots lying in the first quadrant of the λ-plane have the following asymptotic Eqs. :

$$\lambda_k^{(1)} = {}^1/_2 \ln (3\pi + 4k\pi) + i(3\pi/4 + k\pi) + O(k^{-1} \ln k) \quad (3.8)$$

$$\lambda_k^{(2)} = {}^1/_2 \ln (\pi + 4k\pi) + i(\pi/4 + k\pi) + O(k^{-1} \ln k) \quad (3.9)$$

We shall now use the Green's function to invert the operator (3.3). If $G(y, \eta, \lambda)$ is the Green's function of (3.3), then we have by definition

$$u^*(y, \lambda) = \int_{-1}^{1} G(y, \eta, \lambda) \varphi^*(\eta, \lambda) d\eta \quad (3.10)$$

and $G(y, \eta, \lambda)$ has the form

$$G(y, \eta, \lambda) = \frac{1}{4\lambda^3} \Big[\mathrm{sh}\,\lambda\, \mathrm{ch}\,\lambda\, \mathrm{ch}\,\lambda (y + \eta) - \quad (3.11)$$

$$- (1 \pm y \mp \eta) \lambda\, \mathrm{ch}\, \lambda (y - \eta) \pm \mathrm{sh}\, \lambda (y - \eta) - \frac{a^{(1)}(y) a^{(1)}(\eta)}{\mathrm{sh}\,\lambda\, \mathrm{ch}\,\lambda + \lambda} - \frac{a^{(2)}(y) a^{(2)}(\eta)}{\mathrm{sh}\,\lambda\, \mathrm{ch}\,\lambda - \lambda} \Big]$$

where the upper sign applies when $y < \eta$, and the lower one when $y > \eta$, and

$$a^{(1)}(y) = \lambda y \mathrm{sh} \lambda y + \mathrm{sh}^2 \lambda \mathrm{ch} \lambda y, \quad a^{(2)}(y) = \lambda y \mathrm{ch} \lambda y - \mathrm{ch}^2 \lambda \mathrm{sh} y$$

From (3.11) we see, that $G(y, \eta, \lambda)$ is a meromorphic function of λ and that the roots of (3.5), i.e. the eigenvalues of the operator (3.3), are its poles. It can easily be shown that $G(y, \eta, \lambda)$ is bounded on the contours $|\lambda| = |{}^1/_2 \ln \; 4n\pi + in\pi|$ as follows: $|G(y, \eta, \lambda)| \leqslant C|\lambda|^{-1}$, $0 < C < \infty$, hence it can be represented ([2], p. 321) as the sum of its principal parts

$$G(y, \eta, \lambda) = \sum_{k=1}^{\infty} \left[\frac{a_k^{(1)}(y) a_k^{(1)}(\eta)}{8\lambda_k^{(1)3} \mathrm{ch}^2 \lambda_k^{(1)} (\lambda_k^{(1)} - \lambda)} + \frac{a_k^{(2)}(y) a_k^{(2)}(\eta)}{8\lambda_k^{(2)3} \mathrm{sh}^2 \lambda_k^{(2)} (\lambda_k^{(2)} - \lambda)} \right] \quad (3.12)$$

where the right-hand side series converges uniformly in y, η and λ in any finite part of the λ-plane (here and in the following, four terms corresponding to the roots $\lambda_k, -\lambda_k$,

$\bar{\lambda}_k$ and $-\bar{\lambda}_k$ will be denoted by a single index k).

Relations (3.6) to (3.9) imply that this series converges uniformly when λ passes through the whole of the real axis.

Since (3.12) converges uniformly in all parameters, we can insert (3.12) into (3.10) and perform a term by term integration. This yields

$$u^*(y,\lambda)=\sum_{k=1}^{\infty}\Big[\frac{a_k^{(1)}(y)}{8\lambda_k^{(1)3}\operatorname{ch}^2\lambda_k^{(1)}(\lambda_k^{(1)}-\lambda)}\int_{-1}^{1}a_k^{(1)}(\eta)\varphi^*(\eta,\lambda)\,d\eta+$$

$$+\frac{a_k^{(2)}(y)}{8\lambda_k^{(2)3}\operatorname{sh}^2\lambda_k^{(2)}(\lambda_k^{(2)}-\lambda)}\int_{-1}^{1}a_k^{(2)}(\eta)\varphi^*(\eta,\lambda)\,d\eta\Big] \tag{3.13}$$

where the right-side series converges uniformly in y and λ.

Applying the inverse Fourier transform to (3.13) and replacing $\varphi^*(\eta,\lambda)$ with its equivalent given by (3.4), we obtain

$$u_1(x,y)=\frac{1}{\pi}\int_{-\infty}^{\infty}d\lambda\sum_{k=1}^{\infty}e^{i\lambda x}\Big[\frac{a_k^{(1)}(y)}{8\lambda_k^{(1)3}\operatorname{ch}^2\lambda_k^{(1)}(\lambda_k^{(1)}-\lambda)}\times$$

$$\times\int_{-1}^{1}a_k^{(1)}(\eta)\int_{h-\delta}^{h}\varphi(\xi,\eta)\cos\lambda\xi\,d\xi\,d\eta+$$

$$+\frac{a_k^{(2)}(y)}{8\lambda_k^{(2)3}\operatorname{sh}^2\lambda_k^{(2)}(\lambda_k^{(2)}-\lambda)}\int_{-1}^{1}a_k^{(2)}(\eta)\int_{h-\delta}^{h}\varphi(\xi,\eta)\cos\eta\xi\,d\xi\,d\eta\Big] \tag{3.14}$$

Next we shall prove that (3.14) can be integrated term by term. It will be sufficient to show that

$$\int_{-\infty}^{\infty}\sum_{k=n+1}^{\infty}T_k(\lambda)\,d\lambda\to0\qquad\text{as }n\to\infty$$

where $T_k(\lambda)=T_k^{(1)}(\lambda)+T_k^{(2)}(\lambda)$ denotes the kth term under the summation sign in (3.14). Using (3.2), (3.6) and (3.7) we find, that

$$\Big|\int_{-1}^{1}a_k(\eta)\int_{h-\delta}^{h}\varphi(\xi,\eta)\cos\lambda\xi\,d\xi\,d\eta\Big|=\frac{1}{|\lambda|}\Big|\int_{-1}^{1}a_k(\eta)\int_{h-\delta}^{h}\frac{\partial\varphi}{\partial\xi}\sin\lambda\xi\,d\xi\,d\eta\Big|\leqslant$$

$$\leqslant\frac{C|\lambda_k|^{3/2}}{|\lambda|}\int_{-1}^{1}\int_{h-\delta}^{h}\Big|\frac{\partial\varphi}{\partial\xi}\Big|\,d\xi\,d\eta\leqslant C_1\frac{|\lambda_k|^{3/2}}{|\lambda|}$$

from which

$$|T_k(\lambda)|\leqslant|T_k^{(1)}(\lambda)|+|T_k^{(2)}(\lambda)|\leqslant$$

$$\leqslant C_2\Big(\frac{1}{|\lambda|\cdot|\lambda_k^{(1)}|\cdot|\lambda_k^{(1)}-\lambda|}+\frac{1}{|\lambda|\cdot|\lambda_k^{(2)}|\cdot|\lambda_k^{(2)}-\lambda|}\Big)$$

follows.

Since λ lies on the real axis while λ_k have asymptotic equations (3.8) and (3.9), we can easily show that

$$|\lambda_k-\lambda|=|\lambda_k-\lambda|^{1/2}|\lambda_k-\lambda|^{1/2}\geqslant\alpha|\lambda|^{1/2}|\lambda_k|^{1/2}\qquad(\alpha>0)$$

Finally we have

$$|T_k(\lambda)|\leqslant C_3|\lambda|^{-3/2}(|\lambda_k^{(1)}|^{-3/2}+|\lambda_k^{(2)}|^{-3/2})$$

so that

$$\left|\int_{-\infty}^{\infty}\sum_{k=n+1}^{\infty}T_k(\lambda)\,d\lambda\right|\leqslant\left|\int_{-A}^{A}\sum_{k=n+1}^{\infty}T_k(\lambda)\,d\lambda\right|+\left|\left(\int_{-\infty}^{-A}+\int_{A}^{\infty}\right)\sum_{k=n+1}^{\infty}T_k(\lambda)\,d\lambda\right|\leqslant$$

$$\leqslant\int_{-A}^{A}\left|\sum_{k=n+1}^{\infty}T_k(\lambda)\right|d\lambda+2\int_{A}^{\infty}\sum_{k=n+1}^{\infty}|T_k(\lambda)|\,d\lambda\leqslant 2A\varepsilon+8C_3\varepsilon\int_{A}^{\infty}\lambda^{-3/2}\,d\lambda=M\varepsilon$$

$$(n>N)$$

Hence term by term integration is allowed. We have

$$u_1(x,y)=\frac{1}{\pi}\sum_{k=1}^{\infty}\left[\frac{a_k^{(1)}(y)}{8\lambda_k^{(1)3}\operatorname{ch}^2\lambda_k^{(1)}}\int_{-\infty}^{\infty}\frac{e^{i\lambda x}}{\lambda_k^{(1)}-\lambda}\,d\lambda\times\right.$$

$$\times\int_{-1}^{1}a_k^{(1)}(\eta)\,d\eta\int_{h-\delta}^{h}\varphi(\xi,\eta)\cos\lambda\xi\,d\xi+$$

$$\left.+\frac{a_k^{(2)}(y)}{8\lambda_k^{(2)3}\operatorname{sh}^2\lambda_k^{(2)}}\int_{-\infty}^{\infty}\frac{e^{i\lambda x}}{\lambda_k^{(2)}-\lambda}\,d\lambda\int_{-1}^{1}a_k^{(2)}(\eta)\,d\eta\int_{h-\delta}^{h}\varphi(\xi,\eta)\cos\lambda\xi\,d\xi\right]\qquad(3.15)$$

In the following we shall assume that $|x|<h-\delta$. Changing the order of integration in (3.15), we obtain

$$u(x,y)=\frac{1}{\pi}\sum_{k=1}^{\infty}\left[\frac{a_k^{(1)}(y)}{8\lambda_k^{(1)3}\operatorname{ch}^2\lambda_k^{(1)}}\int_{-1}^{1}a_k^{(1)}(\eta)\,d\eta\times\right.$$

$$\times\int_{h-\delta}^{h}\varphi(\xi,\eta)\,d\xi\int_{-\infty}^{\infty}\frac{e^{i\lambda x}}{\lambda_k^{(1)}-\lambda}\cos\lambda\xi\,d\lambda+$$

$$\left.+\frac{a_k^{(2)}(y)}{8\lambda_k^{(2)3}\operatorname{sh}^2\lambda_k^{(2)}}\int_{-1}^{1}a_k^{(2)}(\eta)\,d\eta\int_{h-\delta}^{h}\varphi(\xi,\eta)\,d\xi\int_{-\infty}^{\infty}\frac{e^{i\lambda x}}{\lambda_k^{(2)}-\lambda}\cos\lambda\xi\,d\lambda\right]\qquad(3.16)$$

By [3], we have

$$\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{e^{i\lambda x}}{\lambda_k-\lambda}\cos\lambda\xi\,d\lambda=\begin{cases}-i\exp\left[i\lambda_k(\xi+x)\right] & (\operatorname{Im}\lambda_k>0)\\ i\exp\left[-i\lambda_k(\xi-x)\right] & (\operatorname{Im}\lambda_k<0)\end{cases}\qquad(3.17)$$

Inserting (3.17) into (3.16) and collecting the terms with mutually opposite λ_k, we obtain

$$u(x,y)=2i\sum_{k=1}^{\infty}\left[\frac{a_k^{(1)}(y)\cos\lambda_k^{(1)}x}{8\lambda_k^{(1)3}\operatorname{ch}^2\lambda_k^{(1)}}\int_{-1}^{1}a_k^{(1)}(\eta)\,d\eta\int_{h-\delta}^{h}\varphi(\xi,\eta)\,e^{-i\lambda_k^{(1)}\xi}\,d\xi+\right.$$

$$\left.+\frac{a_k^{(2)}(y)\cos\lambda_k^{(2)}x}{8\lambda_k^{(2)3}\operatorname{sh}^2\lambda_k^{(2)}}\int_{-1}^{1}a_k^{(2)}(\eta)\,d\eta\int_{h-\delta}^{h}\varphi(\xi,\eta)e^{-i\lambda_k^{(2)}\xi}\,d\xi\right](\operatorname{Im}\lambda_k>0)\qquad(3.18)$$

Our previous arguments were valid for any fixed δ. Now we shall investigate the limiting case as $\delta\to 0$. Consider

$$\int_{-1}^{1}a_k(\eta)\,d\eta\int_{h-\delta}^{h}\varphi(\xi,\eta)\,e^{-i\lambda_k\xi}\,d\xi$$

Inserting into it $\varphi(\xi,\eta)$ from (3.2) and integrating by parts in order to free $g(\xi)$

from the derivatives, we obtain

$$\int_{-1}^{1} a_k(\eta)\,d\eta \int_{h-\delta}^{h} \varphi(\xi,\eta)\,e^{-i\lambda_k\xi}d\xi = -\int_{-1}^{1}\Big(\frac{\partial^3 u}{\partial\xi^3} + 2\frac{\partial^3 u}{\partial\xi\,\partial\eta^2} +$$

$$+ i\lambda_k\frac{\partial^2 u}{\partial\xi^2} + 2i\lambda_k\frac{\partial^2 u}{\partial\eta^2} - \lambda_k^2\frac{\partial u}{\partial\xi} - i\lambda_k^3 u\Big)e^{-i\lambda_k\xi}\Big|_{\xi=h-\delta} a_k(\eta)\,d\eta +$$

$$+ \int_{-1}^{1} a_k(\eta)\,d\eta \int_{h-\delta}^{h}\Big(\frac{\partial^4 u}{\partial\eta^4} - 2\lambda_k^2\frac{\partial^2 u}{\partial\eta^2} + \lambda_k^4 u\Big)e^{-i\lambda_k\xi}g(\xi)\,d\xi \qquad (3.19)$$

which yields

$$\lim_{\delta\to 0}\int_{-1}^{1} a_k(\eta)\,d\eta\int_{h-\delta}^{h}\varphi(\xi,\eta)\,e^{-i\lambda_k\xi}d\xi =$$

$$= -e^{-i\lambda_k\xi}\int_{-1}^{1}\Big(\frac{\partial^3 u}{\partial\xi^3} + 2\frac{\partial^3 u}{\partial\xi\,\partial\eta^2} + i\lambda_k\frac{\partial^2 u}{\partial\xi^2} + 2i\lambda_k\frac{\partial^2 u}{\partial\eta^2} -$$

$$-\lambda_k^2\frac{\partial u}{\partial\xi} - i\lambda_k^3 u\Big)\Big|_{\xi=h} a_k(\eta)\,d\eta$$

Passage to the limit under the integral sign in the right-hand side of (3.19) is valid, since by the imbedding theorem ([4], p. 78) all derivatives of u up to the third order inclusive, are continuous. Since uniform convergence of (3.18) in δ is obvious, provided that x does not approach the boundaries $|x| = h$, therefore such a termwise passage to the limit is possible, in which uniform convergence in the remaining parameters will be preserved. Let us introduce the notation

$$c_k^{(1)} = -\frac{ie^{-i\lambda_k^{(1)}h}}{4\lambda_k^{(1)3}\,\mathrm{ch}^2\,\lambda_k^{(1)}}\int_{-1}^{1}\Big(\frac{\partial^3 u}{\partial\xi^3} + 2\frac{\partial^3 u}{\partial\xi\,\partial\eta^2} + i\lambda_k^{(1)}\frac{\partial^2 u}{\partial\xi^2} +$$

$$+ 2i\lambda_k^{(1)}\frac{\partial^2 u}{\partial\eta^2} - \lambda_k^{(1)2}\frac{\partial u}{\partial\xi} - i\lambda_k^{(1)3}u\Big)\Big|_{\xi=h} a_k^{(1)}(\eta)\,d\eta \qquad (3.20)$$

$$c_k^{(2)} = -\frac{ie^{-i\lambda_k^{(2)}h}}{4\lambda_k^{(2)3}\,\mathrm{sh}^2\,\lambda_k^{(2)}}\int_{-1}^{1}\Big(\frac{\partial^3 u}{\partial\xi^3} + 2\frac{\partial^3 u}{\partial\xi\,\partial\eta^2} + i\lambda_k^{(2)}\frac{\partial^2 u}{\partial\xi^2} +$$

$$+ 2i\lambda_k^{(2)}\frac{\partial^2 u}{\partial\eta^2} - \lambda_k^{(2)2}\frac{\partial u}{\partial\xi} - i\lambda_k^{(2)3}u\Big)\Big|_{\xi=h} a_k^{(2)}(\eta)\,d\eta \qquad (3.21)$$

Then

$$u(x,y) = \sum_{k=1}^{\infty}\big[c_k^{(1)}a_k^{(1)}(y)\cos\lambda_k^{(1)}x + c_k^{(2)}a_k^{(2)}(y)\cos\lambda_k^{(2)}x\big] \qquad (\operatorname{Im}\lambda_k > 0) \qquad (3.22)$$

Series (3.22) converges uniformly when $|x| < h$. Moreover, we see from (3.20) and (3.21) that this series can be differentiated in both variables any number of times and, provided that $|x| < h$, the resulting series will remain uniformly convergent.

4. Let us now investigate the behavior of (3.22) on the boundaries $|x| = h$. From (3.20) and (3.21) with (3.6) to (3.9) taken into account, it follows that $c_k = O(e^{-k\pi}\, k^{-5/2})$, therefore on $|x| = h$ the k- x terms of the series will be of the order $k^{-3/2}\,\mathrm{ch}\,\lambda_k y$. Consequently (3.22) converges uniformly when $|x| = h$ and $|y| < 1$. Since for $|y| = 1$

all terms of (3.22) become identically zero, therefore the series also converges at the corners $|x| = h$, $|y| = 1$. Thus we have

$$f(y) = \sum_{k=1}^{\infty} [c_k^{(1)} \cos \lambda_k^{(1)} h a_k^{(1)}(y) + c_k^{(2)} \cos \lambda_k^{(2)} h a_k^{(2)}(y)] \qquad (\operatorname{Im} \lambda_k > 0) \qquad (4.1)$$

which converges uniformly when $|y| < 1$. Let us differentiate (3.22) with respect to x and let us transfer the first n terms to the left-hand side.

$$\frac{\partial u}{\partial x} + \sum_{k=1}^{n} [c_k^{(1)} \lambda_k^{(1)} a_k^{(1)}(y) \sin \lambda_k^{(1)} x + c_k^{(2)} \lambda_k^{(2)} a_k^{(2)}(y) \sin \lambda_k^{(2)} x] =$$

$$= - \sum_{k=n+1}^{\infty} [c_k^{(1)} \lambda_k^{(1)} a_k^{(1)}(y) \sin \lambda_k^{(1)} x + c_k^{(2)} \lambda_k^{(2)} a_k^{(2)}(y) \sin \lambda_k^{(2)} x] \qquad (\operatorname{Im} \lambda_k > 0)$$

Next we shall multiply both sides of this equation by a finite, i.e. infinitely differentiable function $\mu(y)$ which becomes equal to zero together with all its derivatives when $|y| = 1$, and integrate the result with respect to y from -1 to $+1$. Then, a repeated integration by parts of the right-hand side yields

$$\int_{-1}^{1} \left\{ \frac{\partial u}{\partial x} + \sum_{k=1}^{n} [c_k^{(1)} \lambda_k^{(1)} a_k^{(1)}(y) \sin \lambda_k^{(1)} x + c_k^{(2)} \lambda_k^{(2)} a_k^{(2)}(y) \sin \lambda_k^{(2)} x] \right\} \mu(y)\, dy =$$

$$= - \int_{-1}^{1} \mu''(y) \sum_{k=n+1}^{\infty} \{c_k^{(1)} \lambda_k^{(1)} I_y^2 [a_k^{(1)}(y)] \sin \lambda_k^{(1)} x +$$

$$+ c_k^{(2)} \lambda_k^{(2)} I_y^2 [a_k^{(2)}(y)] \sin \lambda_k^{(2)} x\}\, dy \qquad (4.2)$$

where I_y denotes indefinite integration with respect to y. We easily see that the sum under the integral sign in the right-hand side of (4.2), represents the remainder of the series converging uniformly at all x and y, i.e. when $|x| \leq h$ and $|y| \leq 1$. Therefore both sides of (4.2) uniformly tend to zero in x, $|x| \leq h$, as $n \to \infty$. Putting in (4.2) $x = h$ and passing to the limit as $n \to \infty$, we finally obtain

$$\int_{-1}^{1} \left\{ f_1(y) + \sum_{k=1}^{n} [c_k^{(1)} \lambda_k^{(1)} \sin \lambda_k^{(1)} h a_k^{(1)}(y) + \right.$$

$$\left. + c_k^{(2)} \lambda_k^{(2)} \sin \lambda_k^{(2)} h a_k^{(2)}(y)] \right\} \mu(y)\, dy \underset{n\to\infty}{\longrightarrow} 0 \qquad (\operatorname{Im} \lambda_k > 0) \qquad (4.3)$$

for the finite function $\mu(y)$.

5. The method given in Section 3 does not yield explicit expressions for the coefficients c_k in (3.22), since the right-hand sides of (3.20) and (3.21) include the components $u_{\xi\xi\xi}$ and $u_{\xi\xi}$ which cannot be expressed in terms of the boundary conditions (1.17). Coefficients c_k must therefore be obtained by some other method based on the uniqueness of the representation (3.22), which we shall now prove.

The uniqueness of the representation (3.22) is equivalent to the absence of a nontrivial expansion of zero. Let therefore

$$u(x, y) \equiv 0 \equiv \sum_{k=1}^{\infty} [c_k^{(1)} a_k^{(1)}(y) \cos \lambda_k^{(1)} x + c_k^{(2)} a_k^{(2)}(y) \cos \lambda_k^{(2)} x] \qquad (5.1)$$

Assuming $h_1 < h$, let us consider the rectangle $|x| \leq h_1$, $|y| \leq 1$. When $|x| < h$, we can differentiate (5.1) any number of times with respect to both variables, hence

$$u|_{|x|=h_1} = 0, \qquad u_{xx}|_{|x|=h_1} = 0$$

is true.

From this we infer, using the generalized condition of orthogonality of (1.9) from [5], that $c_k \cos \lambda_k h_1 = 0$, i.e. $c_k = 0$ $(k = 1, 2, \ldots)$. So that a nontrivial expansion of a zero is impossible, and this completes the proot ot uniqueness.

BIBLIOGRAPHY

1. Papkovich, P. F., Mechanics of Shipbuilding, Part 2, L., Sudpromgiz, 1941.
2. Markushevich, A. I., Theory of Analytic Functions., M., Gostekhizdat, 1950.
3. Gradshtein, I. S. and Ryzhik, I. M., Tables of Integrals, Sums, Series and Products, M., Fizmatgiz, 1963.
4. Sobolev, S. L., Some Applications of the Functional Analysis in Mathematical Physics, L., Izd. Leningr. Univ., 1950.
5. Grinberg, G. A., On a method proposed by P. F. Papkovich, of solution of the plane problem of the theory of elasticity for a rectangular region, and on the problem of flexure of a rectangular thin plate with two clamped edges and its certain generalizations. PMM Vol. 17, No. 2, 1953.

Translated by L. K.

ROLLING OF ELASTIC BODIES

PMM Vol. 31, No. 5, 1967, pp. 870-876

V. I. MOSSAKOVSKII and I. I. MISHCHISHIN
(Dnepropetrovsk)

(Recived April 25, 1967)

A study is made of the rolling of an elastic cylinder on an elastic foundation. The deformation of the bodies precludes the pure rolling of one body on the other. The rolling is accompanied by sliding. Some recent investigations contain results concerning the rolling of bodies with identical elastic properties. The earliest investigations in this area were conducted by Petrov [1] and Reynolds [2]. This problem was later studied by Fromm [3], who confines himself to the application of Hertz's results [4]. The resistance to rolling of a rigid body on an elastic and inelastic foundation was also investigated by Ishlinskii [5]. The papers of Glagolev [6] and Desoyer [7] contain the general equations for the investigation of the rolling resistance of elastic bodies with different elastic constants. Glagolev solved this problem for bodies with identical elastic constants and examined the limiting case. Desoyer obtained a singular integral equation for the general case and examined this limiting case.

1. Herein, no restrictions are imposed on the elastic properties of the cylinder or the